Two approaches to Fujita's conjecture

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Reference:

Bayer, A., Bertram, A., Macrì, E., Toda, Y.: Bridgeland stability conditions on threefolds II: An application to Fujita's conjecture. J. Algebraic Geom. 23(4), 693–710 (2014).

Conjecture (Fujita)

Let X be a smooth complex projective variety of dimension n and L be an ample divisor on X. Then we have

1. $\mathcal{O}_X(K_X + mL)$ *is globally generated for* $m \geq n+1$.
2. $\mathcal{O}_X(K_X + mL)$ *is very ample for* $m \geq n+2$.

Two proofs of Fujita's conjecture for surfaces

1. Kawamata-Viehweg vanishing theorem + Riemann-Roch;
2. Reider's method: vector bundle technique + Bogomolov's inequality.

The first approach has been generalized to high dimensional case extensively by Siu, Demailly, Ein-Lazarfeld...

Nevertheless, there are difficulties when one generalizes Reider's method to high dimensional varieties.

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Theorem (Reider, Beltrametti and Sommese)

If $|K_S + L|$ is not $(d-1)$-very ample, then there exists an effective divisor $D \subset S$ such that

$$LD - d \leq D^2 < \frac{1}{2}LD < d.$$

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- Assume that there exists a finite subscheme $Z \subset S$ of length d such that

$$e_Z : H^0(S, \mathcal{O}_S(K_S + L)) \to H^0(Z, \mathcal{O}_Z(K_S + L))$$

fails to be surjective.

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fails to be surjective.

- By Kodaira's vanishing, one sees

$$H^1(\mathcal{I}_Z(K_S + L)) = \mathrm{Ext}^1(\mathcal{I}_Z(L), \mathcal{O}_S)^\vee \neq 0.$$

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- There exist a rank two vector bundle E and an exact sequence

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_Z(L) \to 0.$$

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The proof of Reider's theorem

- Since $L^2 > 4d$, by Bogomolov's inequality, E is not μ_H-semistable for any ample divisor H.
- One has an exact sequence $0 \to \mathcal{O}_S(A) \to E \to \mathcal{I}_W(B) \to 0$, where $AH > \frac{1}{2}LH$.
- The composition $A \hookrightarrow E \to \mathcal{I}_Z(L)$ is injective and $D := L - A$ is an effective divisor satisfies the desired inequalities.

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- $E \in \mathrm{Coh}(S)$ is called μ_H-(semi)stable (or slope (semi)stable) if, for all non-zero subsheaves $F \hookrightarrow E$, we have $\mu_H(F) < (\leq) \mu_H(E/F)$.

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Theorem (Bogomolov)

Let E be a μ_H-semistable torsion free sheaf. Then we have

$$\Delta(E) := \mathrm{ch}_1^2(E) - 2\,\mathrm{ch}_0(E)\,\mathrm{ch}_2(E) \geq 0.$$

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- A non-zero class

$$\xi \in H^1(X, \mathcal{I}_Z(K_X + L)) \cong \mathrm{Ext}^2(\mathcal{I}_Z(L), \mathcal{O}_X)$$

gives an exact triangle

$$\mathcal{O}_X[1] \to E^\bullet \to \mathcal{I}_Z(L) \xrightarrow{\xi} \mathcal{O}_X[2].$$

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- a notion of "stability" for $E^\bullet$;
- an inequality of Chern character (involving ch_3) of $E^\bullet$.

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1. Z is additive
2. $\operatorname{Im} Z(E) \geq 0$
3. $\operatorname{Im} Z(E) = 0 \Rightarrow \operatorname{Re} Z(E) \leq 0$

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Lemma

E is μ_H-(semi)stable if for any $0 \neq F \subseteq E$ one has

$$\phi(E) < (\leq) \phi(E/F).$$

Harder-Narasimhan filtration:

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- Every $E \in \mathrm{Coh}(S)$ admits a unique filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that $F_i = E_i/E_{i-1}$ is μ_H semistable and

$$\mu_H^+(E) := \mu_H(F_1) > \mu_H(F_2) > \cdots > \mu_H(F_n) := \mu_H^-(E).$$

Bridgeland stability conditions

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2. $Z : K(\mathcal{A}) \to \mathbb{C}$;

   $0 \neq E \mapsto Z(E) \in \{re^{i\phi\pi} : r > 0(\geq 0), 0 < \phi \leq 1\}$;

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3. Every $0 \neq E \in \mathcal{A}$ has a HN filtration with respect to ϕ;
4. σ satisfies the "support property".

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$(\alpha, \beta) \in \mathbb{R}_{>0} \oplus \mathbb{R}$

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- $\mathcal{T}_\beta := \{E \in \mathrm{Coh}(X) : \mu_H^- > \beta\}$,

  $\mathcal{F}_\beta := \{E \in \mathrm{Coh}(X) : \mu_H^+ \leq \beta\}$;

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- $\mathcal{A}_\beta := \langle \mathcal{T}_\beta, \mathcal{F}_\beta[1] \rangle$;

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  $\mathcal{F}_\beta := \{E \in \mathrm{Coh}(X) : \mu_H^+ \leq \beta\}$;
- $\mathcal{A}_\beta := \langle \mathcal{T}_\beta, \mathcal{F}_\beta[1] \rangle$;
- $Z_{\alpha,\beta} : \mathcal{A}_\beta \to \mathbb{C}$,
  $E \mapsto -H \operatorname{ch}_2^\beta(E) + \frac{1}{2}\alpha^2 H^3 \operatorname{ch}_0(E) + iH^2 \operatorname{ch}_1^\beta(E)$

Theorem (Bridgeland, Arcara-Bertram)

$(\mathcal{A}_\beta, Z_{\alpha,\beta})$ *is a weak stability condition on* X.

Conjecture (Bayer-Macrì-Toda 2014)

For any $Z_{\alpha,\beta}$-stable object $E \in \mathcal{A}_\beta$ with $\operatorname{Re} Z_{\alpha,\beta}(E) = 0$, we have

$$\mathrm{ch}_3^\beta \leq \frac{\alpha^2}{6} H^2 \, \mathrm{ch}_1^\beta(E).$$

Theorem (Bayer-Macrì-Toda 2014)

If BMT's conjecture holds then $Stab(X) \neq \emptyset$.

Theorem (Li, Bernardara-Macrì-Schmidt-Zhao, Piyaratre, Koseki, Bayer-Macrì-Stellari)

BMT's conjecture holds for some Fano 3-folds, Abelian 3-fold, toric 3-folds, quintic 3-folds and some product threefolds in char. zero.

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- S 2020: BMT's conjecture holds for 3-folds with vanishing Chern classes and semistable tangent bundles in any char.
- Schmidt 2017: BMT's conjecture fails for $Bl_P(\mathbb{P}^3)$.

Conjecture (Bernardara-Macrì-Schmidt-Zhao, Piyaratre)

There exists a cycle $\Gamma \in A_1(X)_{\mathbb{R}}$ s.t. $\Gamma H \geq 0$ and for any $Z_{\alpha,\beta}$-stable object $E \in \mathcal{A}_\beta$ with $\operatorname{Re} Z_{\alpha,\beta}(E) = 0$, we have

$$\mathrm{ch}_3^\beta(E) \leq \frac{\alpha^2}{6} H^2 \,\mathrm{ch}_1^\beta(E) + \Gamma \,\mathrm{ch}_1^\beta(E).$$

Theorem (Bernardara-Macrì-Schmidt-Zhao, Piyaratre)

The modified BMT's conjecture holds for Fano 3-folds.

Theorem (Bayer-Bertram-Macrì-Toda)

Assume BMT's conjecture holds for (X, L). Fix a positive integer d. If

then $H^1(X, I_Z(K_X + L)) = 0$ for any zero-dimensional subscheme $Z \subset X$ of length d.

Theorem (Bayer-Bertram-Macrì-Toda)

Assume BMT's conjecture holds for (X, L). Fix a positive integer d. If

1. $L^3 > 49d$;

then $H^1(X, I_Z(K_X + L)) = 0$ for any zero-dimensional subscheme $Z \subset X$ of length d.

Theorem (Bayer-Bertram-Macrì-Toda)

Assume BMT's conjecture holds for (X, L). Fix a positive integer d. If

1. $L^3 > 49d$;
2. $L^2D \geq 7d$ *for every integral divisor class D with* $L^2D > 0$ *and* $LD^2 < d$;

then $H^1(X, I_Z(K_X + L)) = 0$ *for any zero-dimensional subscheme* $Z \subset X$ *of length* d.

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1. $L^3 > 49d$;
2. $L^2D \geq 7d$ *for every integral divisor class* D *with* $L^2D > 0$ *and* $LD^2 < d$;
3. $LC \geq 3d$ *for any curve* $C \subset X$,

then $H^1(X, I_Z(K_X + L)) = 0$ *for any zero-dimensional subscheme* $Z \subset X$ *of length* d.

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1. $L^3 > 49d \Rightarrow E^\bullet$ is not $\nu_{\alpha,\beta}$-semistable for any $0 < \alpha \ll 1$;

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1. $L^3 > 49d \Rightarrow E^\bullet$ is not $\nu_{\alpha,\beta}$-semistable for any $0 < \alpha \ll 1$;
2. (2) and (3) imply that the maximal subobject of $E^\bullet$ is of the form $\mathcal{I}_W(L)$ for some zero-dimensional scheme W. This leads a contradiction.

Corollary

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2. $\mathcal{O}_X(K_X + mL)$ *is very ample for* $m \geq 6$.
3. $\mathcal{O}_X(K_X + mL)$ *is very ample for* $m \geq 5$, *if* $K_X \sim_{num} 0$.

Corollary (S, 2020)

Fujita's conjecture is true for threefolds with semistable tangent bundles and vanishing Chern classes in any char.

Theorem (Langer, 2015)

Let X be a non-uniruled threefold with $K_X \sim_{num} 0$. Then T_X is strongly μ_H-semistable for every ample divisor H.

The classical Bogomolov inequality can be proved by analytic method:

μ_H-stablity of $E \Rightarrow$ existence of Hermitian-Einstein metric on $E \Rightarrow \Delta(E) \geq 0$.

Question

Is there an analytic proof of the BMT conjecture for $E^\bullet := [E_{-1} \xrightarrow{f} E_0]$, where E_i's are vector bundles?

$\nu_{\alpha,\beta}$-semistablity of $E^\bullet \Rightarrow ?$ existence of Hermitian-? metrics on E_i preserved by f $\Rightarrow \mathrm{ch}_3(E^\bullet) \leq ?$.

Thank you!